## Geometry behind linear programming and basics

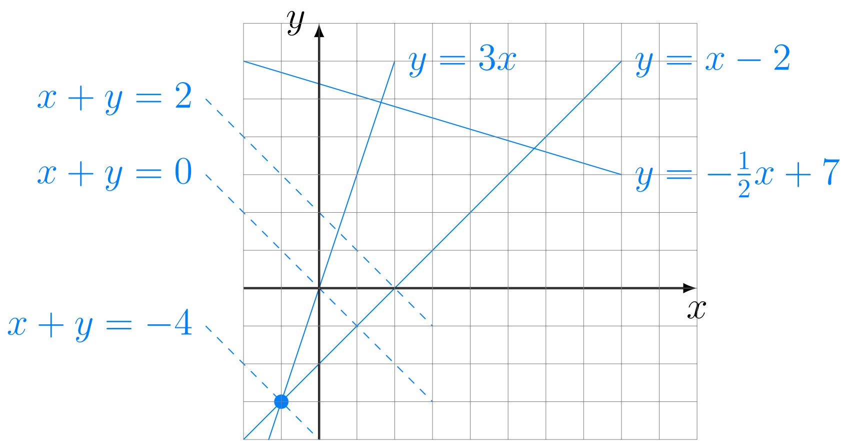
**1:** Solve the following linear program:

$$(LP)\begin{cases}\text{minimize} & x+y\\ \text{s.t.} & x+2y\le 14\\ & 3x-y\ge 0\\ & x-y\le 2,\end{cases}$$

Hint: Plot points $(x, y)$ that satisfy all constraints and then identify the optimal solution among them.

**Solution:** equations:

$$y \le -\frac{1}{2}x+7 \qquad y\le 3x \qquad y \ge x-2$$




Optimum $(x, y) = (-1, -3)$, value of objective function is -4.

Recall that a linear program can be written using a matrix $A \in \mathbb{R}^{m\times n}$ and vectors $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$ as

$$(LP)\begin{cases}\text{minimize} & \mathbf{c}^T\mathbf{x}\\ \text{s.t.} & A\mathbf{x}\le \mathbf{b}\end{cases}$$

**Basic linear programming definitions:**

- *feasible solution* is vector $\mathbf{x}$ such that $A\mathbf{x} \le \mathbf{b}$. In other words, a point satisfying all the constraints.
- *a set of feasible solutions*
- an *optimal solution* is a feasible solution that is maximizing/minimizing the objective function.

**2:** What shape is the set of feasible solutions?

**Solution:** In the example above, a polygon. In general, polyhedra ("unbounded polytope"). In 2D we could say it is an intersection of halfplanes (halfspaces in $xD$).

**3:** What shape is the set of optimal solutions?

**Solution:** In the example above, it is a point. But it can be also a line. Or a special case, where it can be any feasible point (maybe you just want to know if a feasible solution exists).

**4:** Construct a linear program that has no feasible solution.

**Solution:**

$$(LP)\begin{cases}\text{minimize} & x\\ \text{s.t.} & x \le 3\\ & x \ge 4\end{cases}$$

**5:** Construct a linear program that has a feasible solution but does not have an optimal solution.

**Solution:**

$$(LP)\begin{cases}\text{maximize} & x\\ \text{s.t.} & x \ge 4\end{cases}$$

**6:** Construct a linear program that has more than one optimal solution.

**Solution:**

$$(LP)\begin{cases}\text{minimize} & x\\ \text{s.t.} & x \le 3\\ & y \ge 0\\ & y \le 2\end{cases}$$

**Basic geometric definitions (for LP in higher dimension):** Suppose we live in $\mathbb{R}^d$ for some $d \in \mathbb{N}$.

- *hyperplane* is $d-1$ dimensional subspace $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}^T\mathbf{x} = c\}$, where $\mathbf{a} \in \mathbb{R}^d$ and $c \in \mathbb{R}$.
- *closed half space* is $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}^T\mathbf{x} \le c\}$, where $\mathbf{a} \in \mathbb{R}^d$ and $c \in \mathbb{R}$.
- $C \subseteq \mathbb{R}^d$ is *convex* if $\forall \mathbf{x},\mathbf{y} \in C, \forall t \in [0,1], t\mathbf{x} + (1-t)\mathbf{y} \in C$. (line between $\mathbf{x}$ and $\mathbf{y}$ is in $C$)

**7:** Show that the set of feasible solutions to any linear program is a convex set.

**Solution:** We will show that a closed half space is a convex set and that an intersection of convex sets is also a convex set.

Let $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \le c\}$ be a halfspace. We show that $H$ is indeed convex by verifying the definition. Let $\mathbf{x}, \mathbf{y} \in H$ and $t \in [0,1]$.

$$\mathbf{a}^T(t\mathbf{x} + (1-t)\mathbf{y}) = t \cdot \mathbf{a}^T\mathbf{x} + (1-t)\mathbf{a}^T\mathbf{y} \le tc + (1-t)c = c$$

Let $C_1$ and $C_2$ be convex sets. Let $\mathbf{x}, \mathbf{y} \in C_1 \cap C_2$ be arbitrary. Since both $C_1$ and $C_2$ are convex

$$\forall C \in \{C_1, C_2\} \forall t \in [0,1], t\mathbf{x} + (1-t)\mathbf{y} \in C.$$

Hence

$$\forall t \in [0,1], t\mathbf{x} + (1-t)\mathbf{y} \in C_1 \cap C_2.$$

and $C_1 \cap C_2$ is indeed convex.